

Unitary transformation approach for the trapped ion dynamics

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Abstract

We present a way of treating the problem of the interaction of a single trapped ion with laser beams based on successive applications of unitary transformations onto the Hamiltonian. This allows the diagonalization of the Hamiltonian, by means of recursive relations, without performing the Lamb-Dicke approximation.

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I. INTRODUCTION

Trapped ions interacting with laser beams [1,2] have become an extremely interesting system for the investigation of fundamental physics, e.g. the generation of states of the harmonic oscillator [3–5] as well as for potential applications such as precision spectroscopy [6] and quantum computation [7]. The theoretical treatment of the interaction of a trapped ion with one or several laser beams constitutes a complicated problem, being the full Hamiltonian highly nonlinear. Therefore approximations are normally required, such as, for instance, the Lamb-Dicke approximation, in which the ion is considered to be confined within a region much smaller than the laser wavelength. This makes possible to obtain Hamiltonians of the Jaynes-Cummings type, in which the center-of-mass of the trapped ion plays the role of the field mode in cavity QED. Recently [8], it has been suggested a new approach to this problem, based on the application of a unitary transformation which linearizes the total ion-laser Hamiltonian. Moreover, in that approach [8] it is obtained a Jaynes-Cummings type Hamiltonian including counter-rotating terms. A rotating wave approximation (RWA) may be performed provided we are in the special “resonant” regime, or $\Omega \approx \nu$, where Ω is basically related to the laser intensity and ν is the frequency of the ion in the magneto-optical trap. The RWA performed this way allows the exact diagonalization of the transformed Hamiltonian, but on the other hand poses an upper limit to the possible values of the Lamb-Dicke parameter. In this paper we propose a method of diagonalization of the ion-laser Hamiltonian entirely based on unitary transformations. As a first step we linearize the Hamiltonian, as it is done in [8]. The resulting Hamiltonian is further transformed, and cast in a form that it is suitable for diagonalization. Although the diagonalization procedure is an approximate one, based on recursion relations, it is not based in an expansion on any of the parameters of the problem. Therefore, the found solution does not depend on any specific regime, such as the Lamb-Dicke regime, for instance.

This paper is organized as follows: In section 2 we obtain the linearized form of the full Hamiltonian. In section 3 we show how the resulting Hamiltonian may be further

transformed in order to be diagonalized. In section 4 we summarize our conclusions.

II. LINEARIZATION OF THE HAMILTONIAN

We consider a single trapped ion interacting with two laser (classical) plane waves (frequencies ω_1 and ω_2), in a Raman-type configuration ($\omega_L = \omega_1 - \omega_2$). The lasers effectively drive the electric-dipole forbidden transition $|g\rangle \leftrightarrow |e\rangle$ (frequency ω_0), with $\delta = \omega_0 - \omega_L$. We end up with an effective two-level system for the internal degrees of freedom of the atom and the vibrational motion, after adiabatically eliminating the third level in the Raman configuration. This situation is described, in the atomic basis, by the following effective Hamiltonian [9]

$$\hat{H} = \hbar \begin{pmatrix} \nu\hat{n} + \frac{\delta}{2} & \Omega e^{i\eta\hat{X}} \\ \Omega e^{-i\eta\hat{X}} & \nu\hat{n} - \frac{\delta}{2} \end{pmatrix}, \quad (1)$$

being $\hat{X} = \hat{a} + \hat{a}^\dagger$, $\hat{n} = \hat{a}^\dagger\hat{a}$, η the Lamb-Dicke parameter, and \hat{a}^\dagger (\hat{a}) the ion's vibrational creation (annihilation) operator.

By applying the unitary transformation

$$\hat{T} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{D}^\dagger(\beta) & -\hat{D}^\dagger(\beta) \\ \hat{D}(\beta) & \hat{D}(\beta) \end{pmatrix}, \quad (2)$$

to the Hamiltonian in equation (1), where $\hat{D}(\beta) = \exp(\beta\hat{a}^\dagger - \beta^*\hat{a})$ is Glauber's displacement operator, with $\beta = -i\eta/2$, we obtain the following transformed Hamiltonian

$$\hat{\mathcal{H}} \equiv \hat{T}^\dagger \hat{H} \hat{T} = \hbar \begin{pmatrix} \nu\hat{n} + \nu\frac{\eta^2}{4} + \Omega & i\lambda\hat{Y} - \frac{\delta}{\eta\nu} \\ i\lambda\hat{Y} - \frac{\delta}{\eta\nu} & \nu\hat{n} + \nu\frac{\eta^2}{4} - \Omega \end{pmatrix}, \quad (3)$$

where $\hat{Y} = \hat{a} - \hat{a}^\dagger$, and $\lambda = \frac{1}{2}\eta\nu$. This result holds for any value of the Lamb-Dicke parameter η . We have thus transformed our original Hamiltonian in equation (1) into a Jaynes-Cummings-type Hamiltonian, and therefore its exact diagonalization is allowed provided we perform the rotating wave approximation (RWA). However this imposes limitations on the values of the Lamb-Dicke parameter. Because the “effective coupling constant” of the

transformed Hamiltonian is given by $\lambda = \frac{1}{2}\eta\nu$, we shall have $\frac{1}{2}\eta\nu \ll \nu$ if we want to neglect the counter rotating terms (RWA). Moreover, we still have to be “in resonance”, or $\nu \approx 2\Omega$. In this regime the trapped ion-laser system suitably prepared exhibits long-time-scale revivals (superrevivals), as discussed in [8]. Here we are not going to perform the RWA. We propose instead a way of further transforming the Hamiltonian in equation (3) in order to allow its diagonalization in an approximate, although nonperturbative way. In what follows we are going to present a novel method of transforming the Hamiltonian, showing how it may be diagonalized.

III. DIAGONALIZATION OF THE HAMILTONIAN

We have succeeded in transforming the ion-laser Hamiltonian in Equation (2) into a more tractable form (3). Nevertheless, its exact diagonalization has not been achieved yet. The problem is either treated exactly after performing the RWA, or by means of approximate methods such as perturbative expansions or even numerically [10]. Here we propose a different approach to that problem, based on unitary transformations. We are going to restrict ourselves to the resonant case $\delta = 0$. After discarding the constant term $\frac{1}{4}\hbar\nu\eta^2$, which just represents an overall phase factor, we obtain

$$\hat{\mathcal{H}} = \hbar \begin{pmatrix} \nu\hat{n} + \Omega & i\lambda\hat{Y} \\ i\lambda\hat{Y} & \nu\hat{n} - \Omega \end{pmatrix}, \quad (4)$$

Now we define the following unitary transformations

$$\hat{T}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad (5)$$

and

$$\hat{T}_2 = \begin{pmatrix} (-1)^{\hat{n}} & 0 \\ 0 & 1 \end{pmatrix}. \quad (6)$$

These transformations have interesting effects when applied to the Jaynes-Cummings type Hamiltonian in (4). First we apply \hat{T}_1 , which yields

$$\frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \Omega & \lambda \hat{Y} \\ \lambda \hat{Y} & -\Omega \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -i\lambda \hat{Y} & \Omega \\ \Omega & i\lambda \hat{Y} \end{pmatrix}, \quad (7)$$

i.e., the operators are transposed to the diagonal. Then we apply \hat{T}_2 to the matrix which resulted from the operation above, such that

$$\begin{pmatrix} (-1)^{\hat{n}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -i\lambda \hat{Y} & \Omega \\ \Omega & i\lambda \hat{Y} \end{pmatrix} \begin{pmatrix} (-1)^{\hat{n}} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} i\lambda \hat{Y} & \Omega(-1)^{\hat{n}} \\ \Omega(-1)^{\hat{n}} & i\lambda \hat{Y} \end{pmatrix}, \quad (8)$$

where it has been used the fact that $(-1)^{\hat{n}} \hat{a} (-1)^{\hat{n}} = -\hat{a}$ (the same for \hat{a}^\dagger). We have now that \hat{Y} is multiplied by the identity matrix. At this stage we apply \hat{T}_1 again, rearranging the terms in such a way that we obtain a Hamiltonian diagonal in the atomic state basis, or

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} i\lambda \hat{Y} & \Omega(-1)^{\hat{n}} \\ \Omega(-1)^{\hat{n}} & i\lambda \hat{Y} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = i\lambda \hat{Y} + \begin{pmatrix} -\Omega(-1)^{\hat{n}} & 0 \\ 0 & \Omega(-1)^{\hat{n}} \end{pmatrix}. \quad (9)$$

Diagonalization of the total Hamiltonian becomes an easy task now, because the atomic part is already in a diagonal form. The total transformed Hamiltonian may be conveniently expressed in an operator form as

$$\hat{\mathcal{H}}' = \hat{T}_1^\dagger \hat{T}_2^\dagger \hat{T}_1^\dagger \hat{T}^\dagger \hat{H} \hat{T} \hat{T}_1 \hat{T}_2 \hat{T}_1 = \hbar \left(\nu \hat{n} + i\lambda \hat{Y} - \Omega \sigma_z (-1)^{\hat{n}} \right). \quad (10)$$

A general expression for the eigenstates of the Hamiltonian in expression (10) is

$$|\Psi_l\rangle = |\varphi_l^g\rangle |g\rangle + |\varphi_l^e\rangle |e\rangle, \quad (11)$$

with $|\varphi_l^e\rangle = \sum_n C_{n,l}^e |n\rangle$ and $|\varphi_l^g\rangle = \sum_n C_{n,l}^g |n\rangle$. From the eigenvalues equation $\hat{\mathcal{H}}' |\Psi_l\rangle = \Lambda_l |\Psi_l\rangle$ we obtain

$$\left(\nu \hat{n} + i\lambda \hat{Y} - \Omega(-1)^{\hat{n}} \right) \sum_n C_{n,l}^e |n\rangle = \Lambda_l \sum_n C_{n,l}^e |n\rangle, \quad (12)$$

and

$$\left(\nu \hat{n} + i\lambda \hat{Y} + \Omega(-1)^{\hat{n}} \right) \sum_n C_{n,l}^g |n\rangle = \Lambda_l \sum_n C_{n,l}^g |n\rangle. \quad (13)$$

The expansion coefficients (C' 's) may be obtained by means of recursion relations. For instance, for the coefficient $C_{l,n}^e$. From equation (11) we may write

$$\sum_n \left(nC_{n,l}^e |n\rangle + i\lambda C_{n,l}^e \sqrt{n} |n-1\rangle - i\lambda C_{n,l}^e \sqrt{n+1} |n+1\rangle - \Omega(-1)^n C_{n,l}^e |n\rangle \right) = \Lambda_l \sum_n C_{n,l}^e |n\rangle. \quad (14)$$

After rearranging some of the terms, we have the following relation between the coefficients

$$C_{n+2,l}^e = \frac{\sqrt{n+1}}{\sqrt{n+2}} C_{n,l}^e - \frac{i}{\lambda} \frac{[\Lambda + \Omega(-1)^{n+1} - \nu(n+1)]}{\sqrt{n+2}} C_{n+1,l}^e. \quad (15)$$

By multiplying expression (14) by $\langle 0|$, we also obtain the first term

$$C_{1,l}^e = -i \frac{\Omega + \Lambda}{\lambda} C_{0,l}^e. \quad (16)$$

Similar relations may be found for the coefficients $C_{n,l}^g$. Of course the normalization condition $\sum_n |C_{n,l}^g|^2 + \sum_n |C_{n,l}^e|^2 = 1$ should be satisfied. Having diagonalized the transformed Hamiltonian, the evolution of the state vector becomes a trivial task. For that we have to express a generic state $|\psi\rangle$ in terms of the basis states, or

$$|\psi\rangle = \sum_l A_l |\Psi_l\rangle. \quad (17)$$

The choice of a specific initial state determines the set of coefficients A_l . For instance, if we initially prepare the trapped ion in the state $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}|\beta\rangle (|g\rangle - |e\rangle)$, the transformed state will read

$$|\psi(0)\rangle = \hat{T}_1^\dagger \hat{T}_2^\dagger \hat{T}_1^\dagger \hat{T}^\dagger |\Psi(0)\rangle = |0\rangle |e\rangle = \sum_l A_l |\Psi_l\rangle. \quad (18)$$

The coefficients A_l may be determined by multiplying the left hand side of Equation (18) by $\langle n| \langle e|$ and $\langle n| \langle g|$, so that we obtain a set of coupled equations for the coefficients A_l . It is now easy to calculate the time-evolution of the state vector, namely

$$|\psi(t)\rangle = \exp(-i\hat{\mathcal{H}}t/\hbar) \sum_l A_l |\Psi_l\rangle = \sum_l A_l \exp(-i\Lambda_l t/\hbar) |\Psi_l\rangle. \quad (19)$$

The next step is to apply on $|\psi(t)\rangle$ the sequence of transformations $\hat{T}_1^\dagger \hat{T}_2^\dagger \hat{T}_1^\dagger \hat{T}^\dagger$ backwards in order to recover the wanted solution for the state vector which describes the trapped ion system interacting with the laser beams.

IV. CONCLUSIONS

We have presented a novel way of treating the problem of the interaction of trapped ions with laser beams entirely based on unitary transformations. The system Hamiltonian is successively modified and cast in a more tractable form. We propose a method of diagonalization of the transformed Hamiltonian by means of the construction of recursive relations for expansion coefficients in the Fock state basis. Despite of not being an exact diagonalization, there is no need of performing any approximation, such as the Lamb-Dicke limit, for instance. Interesting possibilities for the investigation of such a system in different regimes are opened up, given that there are no restrictions on the relevant parameters and no approximations have been made so far.

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REFERENCES

- [1] Wineland DJ, Monroe C, Itano WM, Leibfried D, King BE, and Meekhof DM, 1998 J. Res. Nat. Inst. Stan. **103**, 259.
- [2] Wieman CE, Pritchard, DE, and Wineland DJ 1999 Rev. Mod. Phys. **71**, S253.
- [3] Heinzen DJ, and Wineland DJ 1990 Phys. Rev. A **42** 2977; Cirac JI, Parkins AS, Blatt R, and Zoller P 1996 Phys. Rev. Lett. **70** 556.
- [4] de Matos Filho RL and Vogel W Phys. Rev. Lett. **76** 608 (1996); de Matos Filho RL and Vogel W 1996 Phys. Rev. A **54** 4560.
- [5] Monroe C, Meekhof DM, King BE, and Wineland DJ 1996 *Science* **272** 1131.
- [6] Wineland DJ, Bollinger JJ, Itano WM, Moore FL, and Heinzen DJ 1992 Phys. Rev. A **46** R6797.
- [7] Cirac JI, Zoller P 1995 Phys. Rev. Lett. **74** 4091.
- [8] Moya-Cessa H, Vidiella-Barranco A, Roversi JA, Freitas Dagoberto S, and Dutra SM 1999 Phys. Rev. A **59** 2518
- [9] Poyatos JF, Cirac JI, Blatt R, and Zoller P 1996 Phys. Rev. A **54** 1532
- [10] de Moura APS 1997 MSc. thesis, Universidade Estadual de Campinas.